Apportioning US Representatives
4 - Paradoxes

The Alabama Paradox

We must return now to the nineteenth century. In the 1880s the chief clerk of the Census Bureau decided to study what would happen if the number of Representatives changed. He computed the apportionment of the House for all sizes from 275 to 350. Let's look at what happens when changing from 299 to 300 Representatives.

1) Open the file "8_Alabama Paradox1882_to be completed", where Hamilton's method is used to apportion the House in 1882.
   Complete the final apportionment in the L column, based on the data already computed (please note that the fractional parts have already been sorted), so that the House has its 299 Representatives.
2) Do the same for a House size of 300 in the other tab of the spreadsheet.
3) What are the differences in the results (look at the O column)? In particular, complete the following sentence:
   Increasing the House size from 299 to 300 would force Alabama to ………………….
   …………………………………… (!) while …………… and ……………… would gain a seat.

This undesirable behavior of Hamilton's method is now called the "**Alabama Paradox**" and it can happen in multiple occasions. For another example, in 1872, if the House size was changed from 270 to 280, Rhode Island, despite a 10-seat increase for the whole House, would drop from 2 Representatives to just 1 !

In the 1880s and 1890s, House sizes were chosen so that the calculations based on Hamilton's method and Webster's method would agree: 325 seats in the 1880s and 356 in the 1890s. But after the census of 1900, new "numbers games" surfaced.

Apportionments based on Hamilton's method were calculated for all House sizes from 350 through 400. From sizes 382 through 391, Maine's delegation switched from 3 to 4 seats and back again five different times. Colorado was to receive 3 seats for all values computed except for a House size of 357, in which case it would receive 2. The Chair of the House Select Committee, no friend of Colorado, proposed a House size of 357. His proposal was defeated.

In fact, so was Hamilton's method, once and for all. At least twenty years after the Alabama paradox had come to light, and after repeated observations of the numerical fluctuations that afflicted various states, **Congress voted to replace Hamilton with Webster in 1902**. The House size was set at 386, so that no state would lose a seat. In 1912 the size was set at 433 (again, the smallest size so that no state would lose a seat).

The New States Paradox

Before we move too fast into the twentieth century, let us return to 1907 and the entry of Oklahoma into the Union. Although Congress had adopted Webster, clerks for the Representatives, and others, examined what would have happened had Hamilton still been in place. They discovered yet another flaw in the method, known as the "**New States Paradox**." In 1907 the House size was 386. According to its population at the time, upon entry into the Union, Oklahoma was entitled to 5 Representatives, bringing the House size to 391. But when the Hamilton apportionment was re-calculated based on the new House size of 391, New York lost a seat to Maine!

4) Open the "9_New States Paradox" file and look especially at the Summary tab. Explain why New York lost a seat and Maine gained one:

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The Population Paradox

1) Compute the apportionment of the following three-states according to Hamilton's method (look at the decimal part of the quotas) with 25 seats to be assigned.

State	Population	Quota	Round down	Final Apportionment (Hamilton)
State A	13 million			
State B	12 million			
State C	112 million			
Total				25

2) Now let's say State A gained 1 million inhabitants and State C gained 2 million. Compute the new apportionment with the same method.

State	Population	Quota	Round down	Final Apportionment (Hamilton)
State A	14 million			
State B	12 million			
State C	114 million			
Total				25

The **population paradox** occurs when one state loses a seat and another state gains a seat, even though the first state's population increased more than the second state's population. It happened to Virginia and Maine in 1900.

No perfect solution

Hill's method tends to favor the smaller states at the expense of the larger ones, since its rounding is based on the geometric mean. Moreover, as any divisor method, it is subject to the Alabama Paradox.

A perfect method would satisfy all the following requirements:

- No state's number of representatives should decrease if the total number of Representatives increases. (Alabama Paradox)
- Every state should get a number of representatives rounded (up or down) to the nearest integer. For example, if a state should receive 3.4 representatives it can receive 3 or 4. (Quota Rule)
- No state with an increasing population can lose a seat to a state with a decreasing population. (Population Paradox)
- The same formula should apply to every state, and the method should not artificially favour large states at the expense of the smaller ones or vice versa.

Michel Balinski

Alas, in 1983, two American mathematicians, Michel Balinski (who worked in France), and Peyton Young, proved that **there is no perfect apportionment method**.

Peyton Young

The Balinski-Young theorem demonstrates that no method can satisfy all requirements: if it does not exhibit any paradox, then the Quota Rule can be broken (as in Jefferson's method), and if it respects the Quota Rule and the Alabama Paradox, then it can have a Population Paradox effect, etc.

What now?

What is the future of the apportionment question? It is difficult to say. On the one hand, there has been no change in the apportionment method since 1941 and, other than a temporary adjustment when Alaska and Hawaii were added to the Union, there has been no change in the size of the House since the 1920s. On the other hand, Congress has the power to change either of these at any time. With each census there may be a discrepancy between the apportionments provided by different methods, prompting the disaffected parties to seek a remedy. And from an academic point of view, research continues to investigate the inherent biases, paradoxical behaviors, and other disadvantages of the various methods. In any case, the apportionment of Congress provides a unique blend of interesting mathematics and fascinating political maneuvering.