Apportioning US Representatives
3 – Hill's Method

In the 1920s, Joseph A. Hill (1860 – 1938), the Chief Statistician of the Census Bureau proposed a new method, whose goal is to **keep the relative disparity between states as low as possible**. Hill consulted Edward V. Huntington (1874 – 1952), a mathematician at Harvard University, who refined the method and called it the **Method of Equal Proportions**.

To better understand how it works, let's imagine we have apportioned all the representatives, but haven't bothered to try to make things fair. Some states will have more than their fair share, while others will have less. Clearly the solution is to transfer delegates from one state to another.

Case Study

Let's look at an example with just two states:

State	Population	Number of representatives	Number of people per representative
State *A*	145,000	15	
State *B*	200,000	24	

The number of people per representatives needs to be as close as possible in each state.

1) Compute these quantities in the last column.

State … is less well represented than state …, and we can look at what happens if we transfer a seat from … to ….

State	Population	Number of representatives	Number of people per representative
State *A*	145,000		
State *B*	200,000		

…………………………… less well represented than …………… and we can look at the transfer of another seat.

State	Population	Number of representatives	Number of people per representative
State *A*	145,000		
State *B*	200,000		

Is the situation better than one step before? To answer that question, Hill's method proposes to look at the **relative difference**: $\frac{\text{larger value} - \text{smaller value}}{\text{smaller value}}$

From ……… to ………, the relative difference between states A and B is

…………………………………………………………

While, from ……… to ……… the relative difference is ………………………………………
The relative disparity is ………… after the transfer, and therefore makes things ………
The ideal situation here is with …… and …… representatives.

General case with 2 states

Let's study the general case, with only two states.

State A (with population A) currently has n representatives and state B (with population B) has m representatives.

The number of people per representatives in each state is …… and ……

Let's say that a representative in state A represents more people than a representative in state B, which means ……………… In that case, we can consider transferring one seat from B to A.

But would the transfer make things better?

After the transfer, A now has ……… representatives, and B now has ……… representatives.

The number of people per representatives in each state is now …… and ……

If, after the transfer, a representative in state A still represents more people than a representative in state B, which means if ………….. , then the transfer does make things better (the disparity is smaller and we can consider another transfer in the same way)

But what if $\frac{A}{n+1} < \frac{B}{m-1}$?

Before the transfer of a seat, the relative difference was …………………………………

After the transfer of a seat, the relative difference is now ………………………………

For the transfer to make things fairer, the relative difference should be lesser than before, meaning ……………………………………………

2) By rearranging, multiplying by the same quantity and by simplifying, prove that it is equivalent to $\frac{A^2}{n(n+1)} > \frac{B^2}{m(m-1)}$

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Taking the square roots of both sides (which is possible since all quantities are positive), we obtain: $\frac{A}{\sqrt{n(n+1)}} > \frac{B}{\sqrt{m(m-1)}}$.

This means that if this criterion is verified, then transferring a seat from B to A is a good thing to reduce the disparity.

First implementation

It is worth noticing that each side is only composed of variables from one state (A and n, then B and m), and we can use that to generalize to a situation with more than two states.

For each state, the quantity $\frac{\textbf{Population}}{\sqrt{\textbf{representatives}(\textbf{representatives}+\mathbf{1})}}$ is called the **priority value**.

The state with the highest priority value should receive a seat from some other state. In practice, we don't have to decide which state that seat comes from, and we can proceed as follows:

- Step 1 : Assign 1 representative to each state
- Step 2 : Compute the priority values $\frac{\text{Population}}{\sqrt{\text{representatives}(\text{representatives}+1)}}$ for each state.
- Step 3: Assign an additional representative to the state with the highest priority value and recompute that state's priority value. (Note the other priority values are unchanged)
- Repeat Step 3 until all desired representatives are assigned.

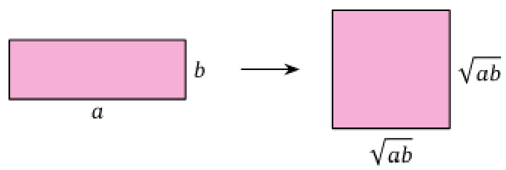
3) Open the file "6_Hill1940_to be completed" and implement the method above until 70 representatives have been assigned. (To make things easier the top priority value should appear in bold)

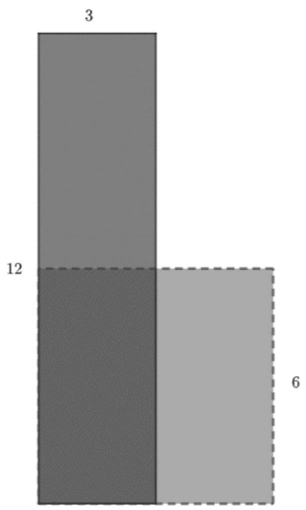
A detour with the Geometric mean

In many cases, we associate the mean of two numbers a and b with the quantity $\frac{a+b}{2}$. This corresponds to the **arithmetic mean**, but this is not the only notion of the mean.

The **geometric mean** of two positive numbers with the same sign is defined as the square root of the product of the two numbers: $\sqrt{\boldsymbol{ab}}$

The geometric mean of the two numbers is the side length of the square that has the same area as the rectangle with side lengths a and b.





For example, the geometric mean of 3 and 12 is $\sqrt{3 \times 12} = \sqrt{36} = 6$

The geometric mean of 5 and 7 is $\sqrt{35}$

Prove that the arithmetic mean is always greater than the geometric mean (begin by squaring on each side): $\frac{a+b}{2} \geq \sqrt{ab}$

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Second implementation

Hill's method has a more global (less step-by-step) implementation than the one explained before. Indeed, it is equivalent to a traditional method of rounding, but **the rounding is now based on the geometric mean**:

- Let $D = \frac{\text{U.S. Population}}{\text{Number of House seats}}$
- Compute State quota $= \frac{\text{State Population}}{D}$ for each state.
- Round this value down and call this rounded number n.
- Compare the *State quota* to the geometric mean of n and $n + 1$, which is $\sqrt{n(n+1)}$
- If the *State quota* exceeds the geometric mean, give the state $n + 1$ representatives. Otherwise, give it n representatives.
- Adjust D (increase or decrease) as necessary to ensure that the total number of seats apportioned agrees with the number of available seats.

A bill finally passed in 1929 requiring the President to send to the Congress apportionments based on Webster, on Hill, and on the method used in the previous apportionment. If Congress took no specific action, the method last used would automatically be employed again. In the 1930s, not only was Webster the last used, but Webster and Hill agreed. This good fortune did not extend to the 1940s.

In 1941 the apportionments as computed by Webster and Hill differed in only two states. Webster gave 18 seats to Michigan and 6 to Arkansas, while Hill gave 7 to Arkansas and 17 to Michigan. It so happened that Michigan was a state that tended to be Republican while Arkansas was Democratic. A representative from Arkansas sponsored a bill to use Hill's method to apportion the House. Every Republican voted against the bill, while every Democrat (except those from Michigan) voted in favor. **The bill passed in 1941 and Hill's method has been used to apportion the seats in a 435 member House ever since**.

1) Open the file "7_Hill2020_to be completed"

- Compute the original D with the formula and put the result in cell G1
- Copy the value in cell J1
- Use the J1 cell to compute each state quota in column C.
- Complete the columns D and E with the instructions ARRONDI.INF et ARRONDI.SUP
- Compute the geometric mean of those two values in column F.
- Compute the apportionment in column G using the following instruction: SI(condition à tester ;valeur si respectée ; valeur si non respectée)
- Adjust the value in cell J1 so that the total number of representatives equals to 435. What is the adjusted D? ……………….